Finding the Angle of a 2-D Distribution

T. Troy Stark
Spiricon Inc.

Part I

Given a distribution, $A(x, y)$, for example an energy distribution of a beam profile or a distribution of centroids on a map –or even a distribution of mass on a 2-D surface– you may want to find the angle that the axis of that distribution makes with a defined coordinate system on the 2-D surface. That angle can be computed from the variances of the distribution. The equations for the variances are given below for a coordinate system coincident with the centroid of the distribution:

X direction variance: $\sigma_x^2 = \dfrac{\iint x^2 A(x, y)dxdy}{\iint A(x, y)dxdy}$, Y direction variance: $\sigma_y^2 = \dfrac{\iint y^2 A(x, y)dxdy}{\iint A(x, y)dxdy}$ and the mixed variance: $\sigma_{xy}^2 = \dfrac{\iint xyA(x, y)dxdy}{\iint A(x, y)dxdy}$

The angle is given by: $\theta = \frac{1}{2} \cdot \arctan\left(\dfrac{2\sigma_{xy}^2}{\sigma_x^2 - \sigma_y^2}\right)$, but there are a few tricks to remember.

You'll notice that, since the arctangent function gives an angle between -90° and +90°, the equation for the angle must return an angle between -45° and +45°. But there are nine (9) possible distribution angles and some of them require some manipulation after the above equation in order to get the correct angle.

We'll start with defining what we mean by the correct angle: It is the angle measured from the coordinate X axis to the major axis of the distribution and this angle is constrained to be between -90° and +90°.

Below are images illustrating the nine possible distributions and some information about the three variances that will help in determining the correct angle using the angle formula given above.

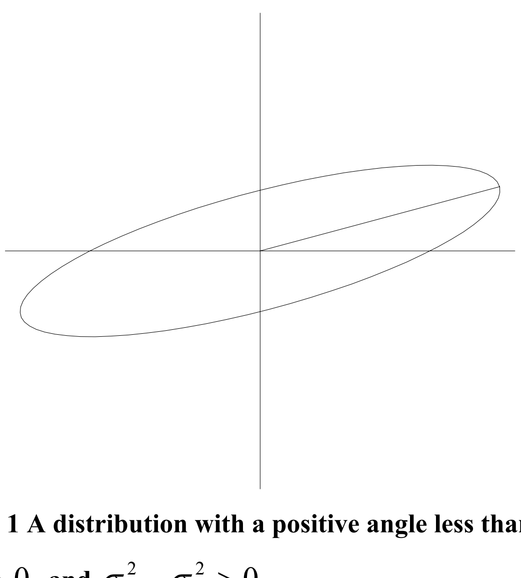

Figure 1 A distribution with a positive angle less than 45°

$\sigma_{xy}^2 > 0$ **and** $\sigma_x^2 - \sigma_y^2 > 0$

The angle formula will give the correct angle in this case.

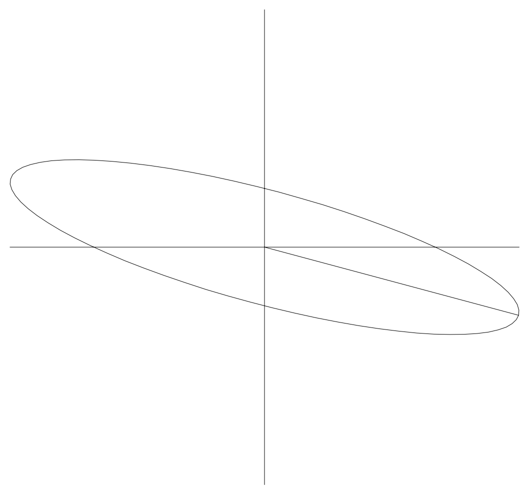

Figure 2 A distribution with a negative angle less than (or rather less negative than) -45°

$\sigma_{xy}^2 < 0$ **and** $\sigma_x^2 - \sigma_y^2 > 0$

The angle formula will give the correct angle in this case.

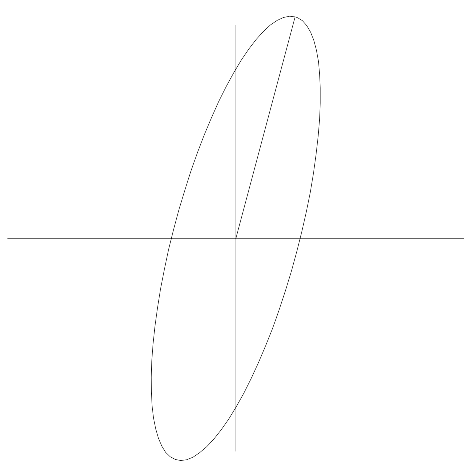

Figure 3 A distribution with a positive angle greater than 45°

$\sigma_{xy}^2 > 0$ **and** $\sigma_x^2 - \sigma_y^2 < 0$

The angle formula will give the negative complementary angle in this case and the correct angle is obtained by adding 90°.

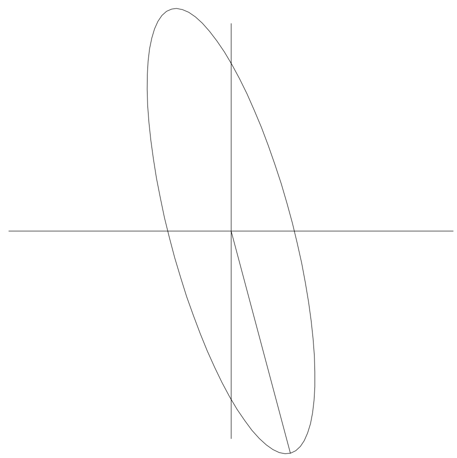

Figure 4 A distribution with a negative angle greater than (or rather more negative than) -45°

$\sigma_{xy}^2 < 0$ **and** $\sigma_x^2 - \sigma_y^2 < 0$

The angle formula will give the positive complementary angle in this case and the correct angle is obtained by subtracting 90°.

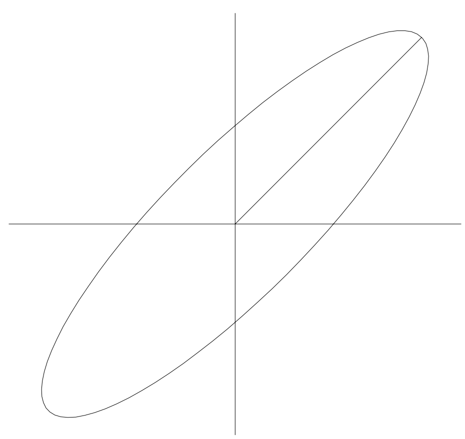

Figure 5 A distribution with an angle of exactly positive 45°

$\sigma_{xy}^2 > 0$ **and** $\sigma_x^2 - \sigma_y^2 = 0$

Depending on the implementation, the angle formula will suffer a divide by zero error or give the correct angle.

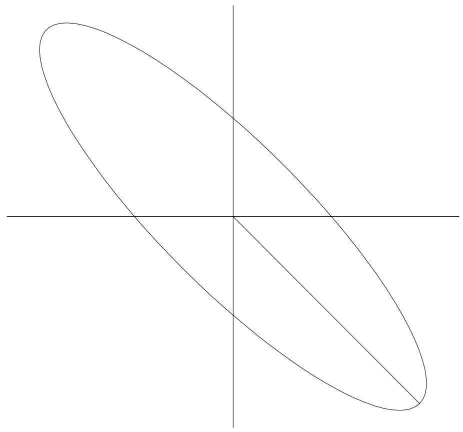

Figure 6 A distribution with an angle of exactly negative 45°

$\sigma_{xy}^2 < 0$ **and** $\sigma_x^2 - \sigma_y^2 = 0$

Depending on the implementation, the angle formula will suffer a divide by zero error or give the correct angle.

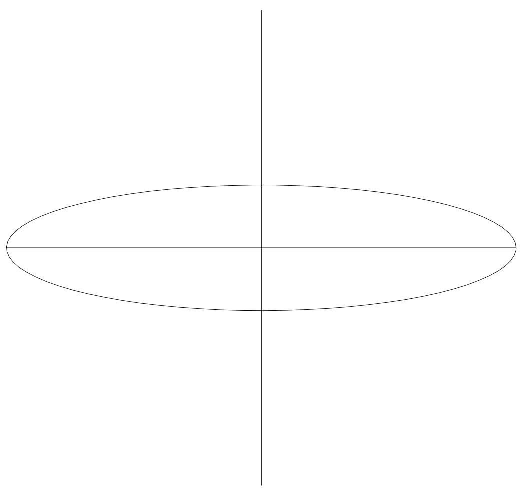

Figure 7 A distribution with an angle of exactly 0°

$\sigma_{xy}^2 = 0$ **and** $\sigma_x^2 - \sigma_y^2 > 0$

The angle formula will usually give the correct angle.

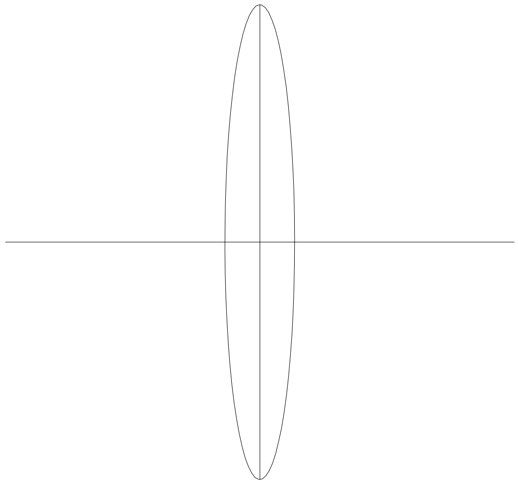

Figure 8 A distribution with an angle of 90°

$\sigma_{xy}^2 = 0$ **and** $\sigma_x^2 - \sigma_y^2 < 0$

The angle formula will usually give the complementary angle and will need to be adjusted. Of course, there is some ambiguity here about whether the angle is +90° or -90° and you will have to make that decision.

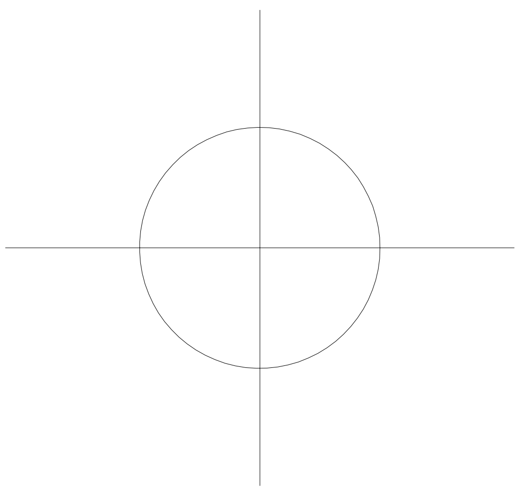

Figure 9 A completely circular distribution with an undefined angle.

$\sigma_{xy}^2 = 0$ **and** $\sigma_x^2 - \sigma_y^2 = 0$

The angle formula cannot be successfully applied in this situation.

With this information, you should be able to successfully determine the angle of any distribution. Note that the first four are the most likely possibilities, since a naturally occurring distribution will rarely have the characteristics of the last five, but it is necessary to accommodate the possibility of these last five in any software program in order to avoid obvious errors or application failures.

Part II

Using An Arbitrary Coordinate System

Recall that the definitions of variances used for calculating the angle of a 2-D distribution, $A(x, y)$, were given in a coordinate system with the origin at the centroid of the distribution.

X direction variance: $\sigma_x^2 = \dfrac{\iint x^2 A(x,y)dxdy}{\iint A(x,y)dxdy}$, Y direction variance: $\sigma_y^2 = \dfrac{\iint y^2 A(x,y)dxdy}{\iint A(x,y)dxdy}$ and the mixed variance: $\sigma_{xy}^2 = \dfrac{\iint xyA(x,y)dxdy}{\iint A(x,y)dxdy}$

In a coordinate system with an origin not so easily moved, such as is the case with data from a camera frame, the equations are:

X direction variance: $\sigma_x^2 = \dfrac{\iint (x-\overline{x})^2 A(x,y)dxdy}{\iint A(x,y)dxdy}$,

Y direction variance: $\sigma_y^2 = \dfrac{\iint (y-\overline{y})^2 A(x,y)dxdy}{\iint A(x,y)dxdy}$

and the mixed variance: $\sigma_{xy}^2 = \dfrac{\iint (x-\overline{x})(y-\overline{y}) A(x,y)dxdy}{\iint A(x,y)dxdy}$

where $\overline{x} = \dfrac{\iint xA(x,y)dxdy}{\iint A(x,y)dxdy}$ and $\overline{y} = \dfrac{\iint yA(x,y)dxdy}{\iint A(x,y)dxdy}$

Now, let's do a little algebra inside the integrals in the numerators:

$$(x-\overline{x})^2 = x^2 + \overline{x}^2 - 2\cdot x\cdot \overline{x}$$

$$(y-\overline{y})^2 = y^2 + \overline{y}^2 - 2\cdot y\cdot \overline{y}$$

$$(x-\overline{x})(y-\overline{y}) = x\cdot y + \overline{x}\cdot \overline{y} - x\cdot \overline{y} - \overline{x}\cdot y$$

A little more algebra and noting the definitions of the resulting integral ratios yields:

$$\sigma_x^2 = \frac{\iint x^2 A(x,y)dxdy}{\iint A(x,y)dxdy} - \overline{x}^2$$

$$\sigma_y^2 = \frac{\iint y^2 A(x,y)dxdy}{\iint A(x,y)dxdy} - \overline{y}^2$$

$$\sigma_{xy}^2 = \frac{\iint xyA(x,y)dxdy}{\iint A(x,y)dxdy} - \bar{x} \cdot \bar{y}$$

Since these definitions, which have only been adjusted for an arbitrary coordinate system, are equivalent to the definitions used in Part I, then we can expect the values of the angle to be calculated from:

$$\theta = \tfrac{1}{2} \cdot \arctan\left(\frac{2\sigma_{xy}^2}{\sigma_x^2 - \sigma_y^2}\right)$$

just as was described in Part I.

There is another possible alteration in the coordinate system that we should consider. Suppose that the y values were measured as increasing as we move down instead of up on our coordinate system. Then there are only two relevant changes in the analysis done above:

$$(x - \bar{x})(\bar{y} - y) = x \cdot \bar{y} - \bar{x} \cdot \bar{y} - x \cdot y + \bar{x} \cdot y$$

$$\sigma_{xy}^2 = \bar{x} \cdot \bar{y} - \frac{\iint xyA(x,y)dxdy}{\iint A(x,y)dxdy}$$

All of the other results remain. This amounts to a "left handed" coordinate system, which shows up in some situations. Of course, we are assuming that we still wish to define our angle as being positive as we move counter-clockwise from the x axis.

Part III

Using Camera Frame Data

There are two important applications for which camera frame data is used to calculate an angle of a distribution. One is the use of a single frame (or averaged frame) for calculating the angle of the energy or power distribution on that frame. This is the case when trying to find the angle of a laser beam profile relative to the camera's reference frame when doing laser beam analysis. The other application is looking at the more abstract distribution of points on a hypothetical frame, each point being determined by a different frame. This is the case when looking at the pointing stability of a laser beam by looking at the spatial distribution of the peaks or centroids of many beam profiles taken over time.

Looking at the first application, we have an energy or power distribution: $I(x,y)$ which is represented by a value for each pixel. In other words, the function, which was spatially continuous over the 2-D surface of the camera detector, has been "sampled" leaving us with a discrete function $I(x,y) \rightarrow A(x_i, y_j)$ or $A_{i,j}$. Our position on the detector is quantized to specific integral multiples of the pixel size: $(x_i, y_j) = (S_x \cdot i, S_y \cdot j)$, which can be even more simplified by using a square pixel such that $S_x = S_y = S$.

We need discrete or quantized versions of the first and second moments of the distribution. We'll take our definitions from Part II because we will be using a fixed coordinate system, which is not centered on the centroid of the distribution. It is also a left-handed coordinate system because we typically use camera frames such that the y direction index increases as we move down the frame. We will also use a unitless form of the equations, realizing that some of our results need to be multiplied by the square pixel size, though the result we are seeking, the angle, is not one of those.

$$\bar{x} = \frac{\sum_i \sum_j i \cdot A_{i,j}}{\sum_i \sum_j A_{i,j}}$$

$$\bar{y} = \frac{\sum_i \sum_j j \cdot A_{i,j}}{\sum_i \sum_j A_{i,j}}$$

$$\sigma_x^2 = \frac{\sum_i \sum_j i^2 \cdot A_{i,j}}{\sum_i \sum_j A_{i,j}} - \bar{x}^2$$

$$\sigma_y^2 = \frac{\sum_i \sum_j j^2 \cdot A_{i,j}}{\sum_i \sum_j A_{i,j}} - \bar{y}^2$$

$$\sigma_{xy}^2 = \bar{x} \cdot \bar{y} - \frac{\sum_i \sum_j j \cdot i \cdot A_{i,j}}{\sum_i \sum_j A_{i,j}}$$

Then the angle formula:

$$\theta = \tfrac{1}{2} \arctan\left(\frac{2 \cdot \sigma_{xy}^2}{\sigma_x^2 - \sigma_y^2} \right)$$

can be applied with the provisions shown in Part I.

Note: C and C++ have built in functions for the arctangent. The correct one to use is the atan() not atan2(). The reason is that the atan2() expects two inputs which it interprets as x and y values of a coordinate point and will give an angle between 0 and π. What you want, is an angle from the arctangent between – π/2 and + π/2 and the angle function will then give you an answer between – π/4 and + π/4 that must be handled as described in Part I.